MATH6031 Lecture 7

BIG Question : WHY ? How to understand mirror symmetry geometrically ?

Two major approaches

① Kontsevich's Homological Mirror Symmetry Conjecture (1994)

HMS Conj

If X and $\check{X}$ are a mirror pair of CY mfds,

then $$D\,\mathrm{Fuk}(X) \cong D^b\,\mathrm{Coh}(\check{X})$$ as triangulated categories

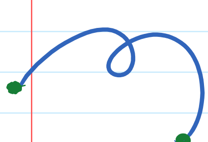
derived Fukaya category of X	derived category of coherent sheaves of $\check{X}$
objects : (twisted complexes of) Lagrangian submfds $L \subset X$	objects : (complexes of) coherent sheaves $\mathcal{E} \downarrow \check{X}$
morphisms : Floer complexes $HF^\cdot(L_1, L_2)$	morphisms : $\mathrm{Ext}^\cdot(\mathcal{E}_1, \mathcal{E}_2)$
A-model of X	B-model of $\check{X}$

Rmks: ① One can also state the HMS conj as

$$\mathrm{Fuk}(X) \simeq D^b_\infty \mathrm{Coh}(\check{X})$$

as an equivalence of A_∞-categories, where $D^b_\infty \mathrm{Coh}(\check{X})$ is a dg-enhancement of $D^b\mathrm{Coh}(\check{X})$.

open strings



② HMS is closely related to the concept of Dirichlet-branes (or D-branes).

② Strominger-Yau-Zaslow Conjecture (1996)

SYZ Conj

If X and $\check{X}$ are a mirror pair of CY mfds, then

(i) X and $\check{X}$ both admit (special) Lagrangian torus fibrations (with sections) to the same base

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & \check{X} \arrow[dl, "\check{\pi}"] \\ & B & \end{tikzcd}$$

← If $\dim_{\mathbb{C}} X = n = \dim_{\mathbb{C}} \check{X}$, then $\dim_{\mathbb{R}} B = n$

(ii) The fibrations π and $\check{\pi}$ are fiberwise dual to each other, i.e. if $\pi^{-1}(b)$ is smooth and isom. to V/Γ, then $(\check{\pi})^{-1}(b)$ is also smooth and is isom to $V^\vee/\Gamma^\vee$

(iii) $\exists$ fiberwise Fourier-type transforms which interchange sympl. geom. data on X w/ cpx geom. data on $\check{X}$, and vice versa.

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§ Mathematics of SYZ

Mclean's Thm (1998)

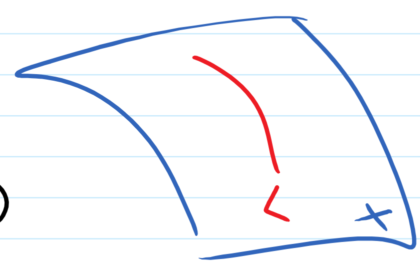
X : CY, $L \subset X$ special Lagr. submfd

ω : sympl. str.

Ω : const. holom. volume form

i.e. $\begin{cases} \omega|_L = 0 \\ \operatorname{Im} \Omega|_L = 0 \quad (\Longleftrightarrow \operatorname{Re}\Omega|_L = \mathrm{vol}_L) \end{cases}$

We consider deform^{ns} of L.



tangent space to the space of deform^{ns} of $L \subset X$ $= \Gamma(L, N_{X/L})$

$$\begin{array}{ccccccccc} 0 \to & TL & \to & TX & \to & N_{X/L} & \to 0 \\ & \wr\wr & & \wr\wr\,\lrcorner\omega & & \wr\wr\,\lrcorner\omega & \\ 0 \to & N^*_{X/L} & \to & T^*X & \to & T^*L & \to 0 \end{array}$$

$$\Rightarrow \quad \Gamma(L, N_{L/X}) \cong \Omega^1(L)$$
$$v \longmapsto \iota_v \omega$$

$\omega|_L = 0 \Rightarrow \iota_v \omega$ is indept of the choice of lift of v.

Similarly, $\operatorname{Im} \Omega|_L = 0$

$\Rightarrow \iota_v \operatorname{Im} \Omega \in \Omega^{n-1}(L)$ is well-defined

<u>A key relation</u>: locally at a pt $p \in L$, we have identifications

$$u : T_p X \xrightarrow{\sim} \mathbb{C}^n \text{ w/ } \overset{\text{coords}}{z_j = x_j + i y_j}$$
$$\omega \longleftrightarrow \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$
$$\Omega \longleftrightarrow dz_1 \wedge \dots \wedge dz_n$$
$$T_p L \longleftrightarrow \mathbb{R}^n \text{ w/ coords } x_j$$

Write $v = \sum_{j=1}^n a_j \frac{\partial}{\partial y_j} \in (N_{L/X})_p$, then we compute

- $(\iota_v \omega)|_L = -\sum_{j=1}^n a_j dx_j$
- $(\iota_v \operatorname{Im} \Omega)|_L = \sum_{j=1}^n \underbrace{(-1)^{j+1}} a_j \underbrace{dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n}$

$$= \sum_{j=1}^n a_j \cdot * dx_j$$

Hodge $*$-operator.

$\Rightarrow \quad \iota_v \operatorname{Im} \Omega = * (-\iota_v \omega)$

Identify a tubular nbh of $L \subset X$ w/

a tubular nbh of the 0-section in $N_{L/X}$.

Then $\exists$ a nbh $U \subset \Gamma(L, N_{L/X})$ of the 0-section

s.t. $\exp_v : L \longrightarrow X$ is well-defined.

Consider

$$F : U \longrightarrow \Omega^n(L) \oplus \Omega^2(L)$$

$$v \longmapsto (\exp_v^*(\operatorname{Im}\Omega), \exp_v^*(\omega))$$

Then the local moduli space of special Lagr. submfds near L is given by $F^{-1}(0)$.

Now $\quad F'(0)(v) = \frac{d}{dt}F(tv)\Big|_{t=0}$

$$= \left(d(\iota_v \operatorname{Im}\Omega)\big|_L, d(\iota_v\omega)\big|_L\right)$$

So $\quad v \in \ker F'(0) \iff \begin{cases} d(\iota_v\omega) = 0 \\ d(\underbrace{\iota_v \operatorname{Im}\Omega}_{= -*\iota_v\omega}) = 0 \end{cases}$ on L

$\iff \quad d(\iota_v\omega) = d^*(\iota_v\omega) = 0$ on L

$\iff \quad \iota_v\omega$ is a harmonic 1-form

Applying the implicit fcn thm gives

Thm (McLean)

The moduli space of special Lagrangian submfd deform$^{\text{ns}}$ of a cpt special Lagrangian $L \subset X$ is a manifold B, whose tangent space at $[L] \in B$ is isom. to $\mathcal{H}^1(L) \cong H^1(L; \mathbb{R})$.

Heuristic reasons behind the SYZ:

- idea of Dirichlet-branes — boundary conditions for open strings.

- X mirror to $\check{X}$ $\Longrightarrow$ $\left\{ \begin{matrix} A\text{-branes} \\ \text{on } X \end{matrix} \right\} \cong \left\{ \begin{matrix} B\text{-branes} \\ \text{on } \check{X} \end{matrix} \right\}$

Now $\check{X} = \begin{matrix}\text{moduli of} \\ \text{pts } p \in \check{X}\end{matrix}$ and each $p \in \check{X}$ is a B-brane (or $\mathcal{O}_p$ skyscraper sheaf) on $\check{X}$

$\Longrightarrow$ $\check{X} = $ moduli of certain A-branes on X

$= \{(L, \nabla)\} / \underline{\text{Ham. isotopy}}$ — isom classes of A-branes on X

where L is a special Lagr. submfd $\subset X$

& ∇ is a flat $U(1)$-conn. over L.

s.t. $(L_p, \nabla_p) \overset{\text{mirror}}{\longleftrightarrow} p \in \check{X}$

So we expect the A-brane (L_p, ∇_p) to behave in the same way as the B-brane $p \in \check{X}$.

- $p \in \check{X}$ covers $\check{X}$ once, i.e. $\check{X} = \bigsqcup_{p \in \check{X}} \{p\}$

$\Longrightarrow$ $\underline{L_p}$ also swap X once

i.e. $X = \bigsqcup L_p$

- Mclean's Thm $\Rightarrow$ deform$^{\text{ns}}$ of L_p is unobstructed and modeled on $H^1(L_p; \mathbb{R})$.

On the other hand, deform$^{\text{ns}}$ of $\nabla_p = H^1(L_p; \mathbb{R}/\mathbb{Z})$

So $b_1(L_p) = \dim_{\mathbb{C}}$ of deform$^{\text{n}}$ space of (L_p, ∇_p)

$= \dim_{\mathbb{C}}$ of deform^n space of $p \in \check{X}$

$= n$

Also we expect $\mathrm{Ext}^\bullet(\mathcal{O}_p, \mathcal{O}_p) = \Lambda^\bullet T_p \check{X}$ (perhaps by HMS)

$$\Rightarrow \quad \underset{\substack{\| \\ H^\bullet(L_p;\mathbb{R})}}{HF^\bullet(L_p, L_p)} = \Lambda^\bullet H^1(L_p;\mathbb{R})$$

So we believe that L_p = torus

Hence we deduce that X admits a sLagr. torus fibration $\begin{array}{c} X \\ \downarrow \pi \\ B \end{array}$

$$\Rightarrow \quad \check{X} = \{(L_p, \nabla_p)\}/\sim$$

$$= \coprod L_b^*$$

$\because$ L_p^* = dual of the torus L_p

$= \{$flat $U(1)$-conn over $L_p\}/\sim$

$\rightsquigarrow$ $\check{X}$ admits the fiberwise dual fibration $\begin{array}{c} \check{X} \\ \downarrow \check{\pi} \\ B \end{array}$.

Consequences of SYZ

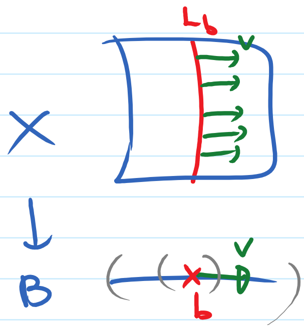
SYZ suggests a way to construct mirrors geometrically.

Suppose X admits a special Lagr. torus fibration without singular fibers:

$$\begin{array}{ccc} X & \supset & L_b \\ \downarrow \pi & & \downarrow \\ B & \ni & b \end{array}$$

Then there are 2 induced ($\mathbb{Z}$-) affine structures on the base:

① Choosing a (locally constant) basis $\underbrace{\gamma_1, \dots, \gamma_n}$ of $H_1(L_b)$, we define 1-forms $\omega_j \in \Omega^1(B)$ by



$$\omega_j(v) = -\int_{\gamma_j} \iota_v \omega \quad \text{for } v \in T_b B$$

$$d\omega = 0 \implies d\omega_j = 0 \quad \text{for } j = 1, \dots, n$$

So locally we have coordinates $\{x_1, \dots, x_n\}$ s.t. $\omega_j = dx_j$.

change of basis of $H_1(L_b)$ $\longrightarrow$ transition maps $\in GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

i.e. $\mathbb{Z}$-affine linear transformations

This defines the symp. ($\mathbb{Z}$-) affine structure on B.

② Similarly, choosing a (locally const) basis $\Gamma_1, \dots, \Gamma_n \in H_{n-1}(L_b)$, we define 1-forms $\lambda_j \in \Omega^1(B)$ by

$$\lambda_j(v) = \int_{\Gamma_j} \iota_v \operatorname{Im} \Omega \quad \text{for } v \in T_b B$$

$$d(\operatorname{Im} \Omega) = 0 \implies d\lambda_j = 0 \quad \text{for } j = 1, \dots, n$$

$\longrightarrow$ this gives local coordinates $\xi_1, \dots, \xi_n$ s.t. $d\xi_j = \lambda_j$ for $j = 1, \dots, n$.

change of basis in $H_{n-1}(L_b)$ $\implies$ transition maps $\in GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

This defines the complex ($\mathbb{Z}$-) affine structure on B.

Rmk : SYZ says that these two affine structures are interchanged under mirror symmetry.

Now (by further assuming that $\pi : X \to B$ admits a Lagr. section), a thm of Duistermaat says that there are global angle coordinates $u_1, \ldots, u_n$ on fibers of π

s.t. $$X \cong T^*B / \Lambda^\vee \quad \text{as sympl. mfds.}$$

where • $\Lambda^\vee \subset \{T^*B\}$ is the lattice subbundle gen. (over $\mathbb{Z}$) by $\{dx_1, \ldots, dx_n\}$

($\{x_1, \ldots, x_n\}$ are the sympl. affine coord. or the action coordinates)

• T^*B is equipped with the canonical sympl. str.
$$\omega_0 = \sum_{j=1}^n dx_j \wedge du_j$$
which descends to $T^*B/\Lambda^\vee$.

In this case, SYZ suggests that the mirror is given by the fiberwise dual:

$$\begin{tikzcd}
X = T^*B/\Lambda^\vee \arrow[dr, "\pi"'] & & \check{X} = TB/\Lambda \arrow[dl, "\check{\pi}"] \\
& B &
\end{tikzcd}$$

fiber $= T_b^*B/(\Lambda^\vee)_b$

dual fiber $= T_bB/\Lambda_b$

We call $\check{X} := TB/\Lambda$ the semi-flat SYZ mirror of X

Here $\Lambda \subset TB$ is the lattice gen. by $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$.

Note: In general, the tangent bundle of a sm. mfd is NOT a complex manifold.

However, TB/Λ is a complex manifold because B is $\mathbb{Z}$-affine; more precisely, if $y_1, \ldots, y_n$ are the fiber coordinates dual to $u_1, \ldots, u_n$, then

$$z_j := \exp(x_j + i y_j), \quad j = 1, \ldots, n$$

are the holom. coordinates on $\check{X} := TB/\Lambda$.